Existence and Uniqueness Theorems for Infinitesimal Micromonad of an Initial Standard Point and Legendre Equation

Ibrahim O. Hamad[1] and Mardan A. Pirdawood[2]

[1]Department of Mathematics, Salahaddin University – Erbil, Karkuk Road, Hawler (Erbil), Kurdistan Region - Iraq

[2]Department of Mathematics, Faculty of Science and Health, Koya University, Kurdistan Region – Iraq

Abstract—In this paper, we use some non-standard concepts to study the analyticity near the singularity. We analyzed and proved the existence and uniqueness theorems for first-order ordinary differential equations in a subset of the monad of the initial standard point. Then, the solutions of the second-order ordinary differential equation (Legendre Equation) are introduced around the singularity in the monad of zero using power series method with suitable transformations for singular points.

***Index Terms*—Existence, Legendre equation, Non-standard analysis, S-continuity, Singularity, Uniqueness theorems.**

I. Introduction

The theory of ordinary differential equations is the most important to the applied mathematics, for example, in the applications of the Natural Sciences, notably physics, and other fields. In this paper, we elucidate the fundamental problems of the existence and uniqueness theorems of a first-order differential equation

$$\frac{dy}{dx} = f(x,y) \text{ with } y(0) = 0$$

Where x and y are in ζ-*Microm*(t), ζ is an infinitesimal, and $t = (x_0, y_0)$ of which x_0 and y_0 are standards, and we are trying to found the solution in the Monad of the singularity for the Legendre equation. The existence and uniqueness theorems of a solution for a first-order differential equation, given a set of initial conditions, are one of the most fundamental results of ODE. The extension of these results is going back to Müller [1], Kamke [2], and Walter [3]; we note that there are various other formulations for producing a proof of the existence and uniqueness theorems for the first-order ODEs in an infinitesimal Micromonad of an initial standard point using some non-standard mathematical analysis tools and Picard's iteration method.

We try to give the most important preface about the main basic statements which make the inclusion of the paper more understandable.

Pure and Applied Science Conference | Koya University
Paper ID: ICPAS2018.PMA66, 6 pages
DOI: 10.14500/icpas2018.pma66
Received 06 March 2018; Accepted 16 March 2018
Conference paper: Published 01 August 2018
Conference track: Pure Mathematics (PMA)
Corresponding author's e-mail: mardan.ameen@koyauniversity.org

II. Principles and Some Definitions of Non-standard Analysis

The following definitions and notations of non-standard analysis will be needed in the sequel.

A real number x is called infinitesimal in case $|x| \leq 1/\nu$ for some nonstandard natural number ν since a non-standard ν is bigger than every standard n, as shown in Fig. 1.

Therefore, if x is an infinitesimal, then $|x| < 1/n$ for every standard natural number n. A real number x is called limited in case $|x| \leq 1/n$ for some standard natural number n [4]. A real number ω is called infinitely large or unlimited if its absolute value $|\omega|$ is larger than any standard integer n [5], as shown in Fig. 2.

Let a and b be two real numbers with $a-b$ is infinitesimal, then we say that a is infinitely close to b, and we write $a \cong b$ [6]. A real number x is called appreciable, if x is limited and it is not infinitesimal [7]. Let r be a limited real number, then the set of all real numbers which are infinitely close to r is called the Monad of r and denoted by m(r) [8]. Fig. 3. Let E be a subset of the standard topological space X. Then, we define the shadow of E, denoted by E, as

$E = {}^{st}\{x \in X: y \cong x \text{ for some } y \text{ in } E\}$

The ζ – Micromonad of zero with a real infinitesimal $\zeta > 0$ is defined as

ζ – Microm (0) = $\{x \in R: x \leq \zeta^n, \forall^{st} n\}$

A standard function f is continuous at a standard point x if for all y, $y \cong x$, then $f(y) \cong f(x)$. A function f is called S-continuous at x if for all y, $y \cong x$, then $f(y) \cong f(x)$ [8]. Hence, every continuous function is not S-continuous function, for example, let $\xi > 0$ be infinitesimal, and let $(x) = \dfrac{\xi}{\xi^2 + x^2}$.

Then, f is continuous at $x = 0$ (this internal property in classical mathematic is true $\forall \xi > 0$), but it is not S-continuous at $x = 0$, because if we choose $y = \xi$, then $f(\xi)$ - $f(0)$ well become unlimited.

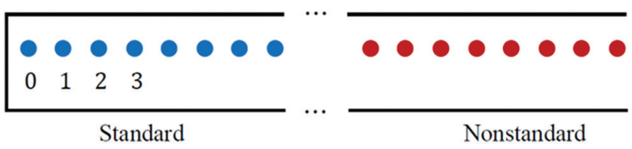

Fig. 1. Structure of extended natural numbers.

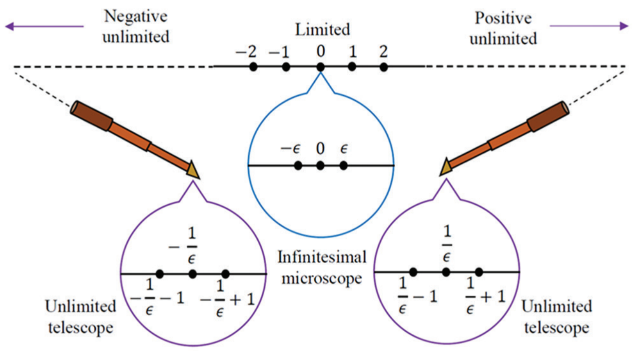

Fig. 2. Structure of extended real numbers, where ∈ is an infinitesimal.

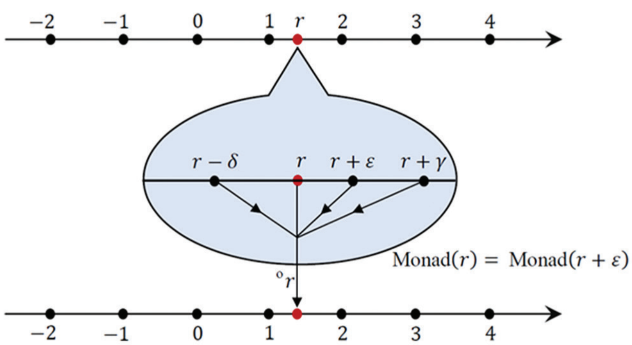

Fig. 3. The monad of r, where δ, γ, and ϵ are infinitesimals.

Theorem 1. [7] Let α be a limited real number and ξ and ζ any two infinitesimal numbers, then:

i. $\xi \bullet \zeta$ is an infinitesimal.
ii. $\xi \bullet \alpha$ is an infinitesimal.
iii. $\xi + \alpha$ is limited.
iv. $\xi + \zeta$ is an infinitesimal (in general, the sum of arbitrary finite number of infinitesimal numbers is an infinitesimal).

Theorem 2: [9] Suppose X and Y are metric spaces with X is compact. Suppose that f:$^{o}X \to {}^{o}Y$: Is an internal, S-continuous function. Suppose that $f(p) \in Yns$ for each $p \in X$. Define F: $X \to Y$ by $F(p) = st(f(p))$. Then, F is continuous and $F(p) \cong f(p)$ for all $p \in {}^{o}X$.

Theorem 3: (Principal of external induction) [10] If E is an internal or external property such that E (0) is true and $E(n) \to E(n+1)$ is true $\forall n \in N$, then $E(n)$ is true $\forall n \in N$.

Lemma 4: (Cauchy principal lemma) [7] If P is any internal property and if P (n) holds for all standard n, then there exist an unlimited $\omega \in N$ such that P (n) is held for all $n \le \omega$.

Now, we define a definition of a set bounded by numbers as follows:

Definition 5: Let X be any standard metric space and for any limited real numbers $x_1, x_2, \ldots, x_n$ for $n \in N$. We denote the interior of the bounded set by $IntBnd$ $(x_1, x_2, \ldots, x_n)$ and is defined as follows:

$IntBnd$ $(x_1, x_2, \ldots, x_n) = \{(y_1, y_2, \ldots, y_n) \in X_n: |y_i| \le x_i$ for $i = 1, \ldots, n\}$

Remark: If we have $x_1, x_2, \ldots, x_n$ which are real infinitesimals, then $IntBnd$ $(x_1, x_2, \ldots, x_n) \subset m$ $(x_1, x_2, \ldots, x_n)$.

III. Existence and Uniqueness Theorems

Given a first-order differential equation:

$$\frac{dy}{dx} = f(x,y) \text{ with } y(0) = 0 \qquad (1)$$

If x and y are in ζ – $Microm$ (t), where ζ is an infinitesimal and $t = (x_0, y_0)$ for x_0 and y_0 are standards. Then we can do a suitable transformation so that we have to assume that x and y are at (0,0), we mean that there is no loss of such generality if the initial value of y is zero. To see this, suppose that instead of equation (1), we have

$$\frac{dy}{dx} = f(x, y(x)) \text{ with } y(x_0) = y_0.$$

We may transform this such as the form of the equation (1). To do that, let $\phi(x) = y(x) - y(x_0)$, note that $\phi(x_0) = y(x_0) - y(x_0) = 0$ and $\frac{d\phi}{dx} = \frac{dy}{dx}$. Therefore, $\frac{d\phi}{dx} = f(x, \phi(x) + y(x_0)) = f_1(x, \phi(x))$. Similarly, for the variable x.

A. Existence Theorem

Suppose that f (x,y) is S-continuous function defined on $IntBnd$ (γ,κ), where γ and κ are real positive infinitesimals. Then, there exists $x \in \gamma$ – $Microm$ (0) so that $y = \psi(x)$ is a solution of equation (1).

B. Uniqueness Theorem

Suppose that both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are S-continuous functions defined on $IntBnd$ (γ,κ), where γ and κ are real positive infinitesimals. Then, there exists $x \in \gamma$ – $Microm$ (0) so that the solution $y = \psi(x)$ was guaranteed by existence theorem which is the unique solution of equation (1).

Now, assume that we have the differential equation $x' = v(x,t)$, then we mean by Picard's mapping is the following mapping of the function ϕ: $t \to x$ into F_ϕ: $t \to x$, to see that the solution to this problem is given as follows:

$$F_f(t) = F_{f_0}(t) + \int_{x_0}^{t} v(f(\tau),\tau)\,d\tau$$

The Picard's iteration method process consists of obtaining a sequence of functions which will get infinitely close to the desired solution. The process will be as follows:

i. Set $F_{\phi_0}(t) = x_0 \forall t$,

ii. $F_{\phi_{n+1}}(t) = x_0 + \int_{x_0}^{t} v(\phi_n(\tau),\tau)\,d\tau$, for $n \ge 1$.

Inspired by the contraction mapping theorem, now assume that the Picard's iteration method (or successive Picard approximations) ϕ, F_{ϕ_1}, F_{ϕ_2}, …, beginning, say, with $\phi = x_0$ [11].

DOI: http://dx.doi.org/10.14500/icpas2018

For example Let $x' = x$, with initial condition $x(t_0) = x(0) = 1$. Hence, the Picard's iteration method can be written as follows:

$\phi = x_0$

$$F_{\phi_1} = x_0 + \int_0^t x_0 \, d\tau = x_0 \, (1+t),$$

$$F_{\phi_2} = x_0 + \int_0^t x_0 \, (1+\tau) d\tau = x_0 \, (1+t+\frac{t^2}{2!}),$$

$$F_{\phi_\omega} = x_0 \left(1+t+\frac{t^2}{2!}+\ldots+\frac{t^\omega}{\omega!}\right) \cong e^t x_0,$$

Where ω is positive unlimited. Thus, e^t is the exact solution of the differential equation $x' = x$ satisfying the initial condition $x(0) = 1$.

C. Proof of Existence and Uniqueness Theorems

To prove existence theorem, we will use Picard's iteration method to construct a sequence of functions $\{\psi_n(x)\}_{n \in N}$. Set $\psi_0(x) = 0$ and $\psi_n(0) = 0$, $\forall n \in N$ and

$$\psi_n(x) = \psi_0(x) + \int_0^x f\left(s, \psi_{n-1}(s)\right) ds = \int_0^x f\left(s, \psi_{n-1}(s)\right) ds \quad (2)$$

Now, we want to prove that $\psi_n(x)$ exists and S-continuous for all $n \in N$. In classical mathematics, every continuous function defined on a compact metric space onto the usual metric space is limited. Since $f(x, y)$ is S-continuous function then by theorem (2), we get that $f(x, y)$ is limited and by definition of limited number there exists a standard natural number μ such that

$|f(x,y)| \leq \mu, \forall (x,y) \in IntBnd(\gamma,\kappa)$.

To show that $\psi_n(x) \in IntBnd(\kappa)$, we restrict $x \in [-\tau, \tau]$ for $\tau < \min\{\frac{\kappa}{\mu}, \gamma\}$ and $\tau \in \gamma - Microm(0)$, then

$$\psi_n(x) = \int_0^x f\left(s, \psi_{n-1}(s)\right) ds$$

Is at most $\mu|x|$ and $\mu|x| \leq \mu\tau < \kappa$ (i.e., $|\psi_n(x)| < \kappa$), as shown in Fig. 4.

To show that $\psi_n(x)$ exists for all $x \in \gamma$–$Microm(0)$, where n is unlimited. We can write $\psi_n(x)$ as

$$\psi_n(x) = \sum_{i=1}^{n} (\psi_i(x) - \psi_{i-1}(x))$$

For $n = 0$, we have $\psi_0(x) = 0$. Now, for any $x \in \gamma - Microm(0)$ and positive real appreciable r such that $r \ll 1$. We will show that

$$|\psi_i(x) - \psi_{i-1}(x)| < r^i, \ \forall \ i \in N$$

Now, applying Theorem (3) on i, where i is limited. Hence, for $i = 0$, it is clear and we have

$$\psi_i(x) - \psi_{i-1}(x) = \int_0^x (f(s, \psi_{i-1}(s)) - f(s, \psi_{i-2}(s)) ds.$$

Apply the mean value theorem to the function $g(y) = f(s,y)$, where the two points are $\psi_{i-1}(s)$ and $\psi_{i-2}(s)$, then there exists $\psi_i(s) \in (\psi_{i-1}(s), \psi_{i-2})(s))$ such that

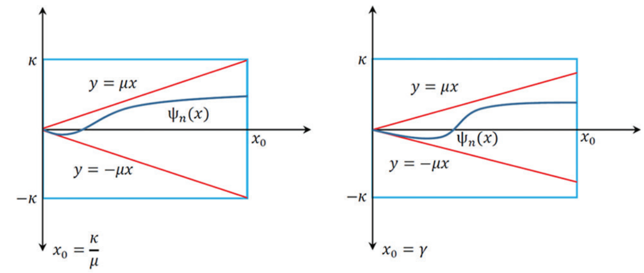

Fig. 4. $|\psi_n(x)| < \kappa$

$g(\psi_{i-1}(s)) - g(\psi_{i-2}(s)) = g'(\psi_i(s)) \bullet (\psi_{i-1}(s) - \psi_{i-2}(s))$,

Where

$$g'(y) = \frac{\partial f}{\partial y} \text{ then}$$

$$\psi_i(x) - \psi_{i-1}(x) = \int_0^x \frac{\partial f}{\partial y} \cdot (\psi_{i-1}(s) - \psi_{i-2}(s)) ds.$$

By assumption of external induction, we have

$|\psi_{i-1}(x) - \psi_{i-2}(x)| < r^{i-1}$,

Where $\frac{\partial f}{\partial y}$ is defined on $IntBnd(\gamma,\kappa)$ and it is limited, then there exists a standard real μ_1 such that

$$|\frac{\partial f}{\partial y}| \leq \mu_1, \forall y \in IntBnd(\kappa).$$

Hence,

$$|\psi_i(x) - \psi_{i-1}(x)| < \int_0^x \mu_1 r^{i-1} \, ds \leq \mu_1 |x| r^{i-1}.$$

Since $x \in \gamma - Microm(0)$. Hence, $\mu_1 \bullet |x|$ is an infinitesimal by theorem (1) (ii), then $\mu_1 \bullet |x| < r$. Therefore,

$|\psi_i(x) - \psi_{i-1}(x)| < r^i$.

Consequently, since it is true for all standard natural number i, then by Cauchy Principle Lemma(4) there exists an unlimited $\omega \in N$ such that $|\psi_i(x) - \psi_{i-1}(x)| < r^i$ for all $i \leq \omega$. Hence,

$$\psi_\omega(x) = \sum_{i=1}^{\omega} (\psi_i(x) - \psi_{i-1}(x)),$$

Moreover,

$$\sum_{i=1}^{\omega} r^i \cong (\frac{1}{1-r} - 1).$$

Hence, by comparison test, we have $\psi_\omega(x)$ exists. Thus, $\psi_n(x)$ exists $\forall x \in \gamma - Microm(0)$ as $n \cong \omega$ where ω is unlimited.

Now, to show that $\psi(_x) = \psi_\omega(x)$ is S-continuous. Let $x_1 \in \gamma - Microm(0)$ if $x_1 \cong x_2$, then we have to show that $\psi(x_1) \cong \psi(x_2)$. Since

$$\psi(x_2) - \psi(x_1) = \psi_\omega(x_2) - \psi_\omega(x_1) = \int_{x_1}^{x_2} f(s, \psi_{\omega-1}(s)) ds.$$

However, for all $s \in IntBnd(\gamma)$, we have $f(s, \psi_{\omega-1}(s))$ is limited. Hence, there exists a standard real number μ such that

$$|\psi(x_2) - \psi(x_1)| \leq \int_{x_1}^{x_2} \mu \, ds \leq \mu |x_1 - x_2|$$

DOI: http://dx.doi.org/10.14500/icpas2018

Given x_1 is infinitely close to x_2 and μ is standard, we get $\mu|x_1-x_2|$ is an infinitesimal. Thus, $\psi(x_2)\cong\psi(x_1)$.

To show that $\psi(x) = \psi_\omega(x)$ satisfies equation (1), we have to show that $\frac{d\psi}{dx} = f(x,\psi(x))$. Now,

$$\psi(x) = \psi_\omega(x) = \int_0^x f(s,\psi_{\omega-1}(s))ds$$

$$== \int_0^x f(s,\psi(s))ds$$

Hence, by fundamental theorem of calculus, we have $\psi'(x) = f(x,\psi(x))$.

Finally, to prove uniqueness theorem, assume that there is another solution $\phi(x)$ satisfies equation (1). Now,

$$\psi(x)-\phi(x)=\int_0^x (f(s,\psi(s))-f(s,\phi(s)))ds$$

If ψ and ϕ are distinct, then there exists a standard $\xi_0>0$ such that for some $x\in\gamma$ – *Microm* (0), we have

$$|\psi(x)-\phi(x)|>\xi_0 \tag{3}$$

Now, let $m = max\{|\psi(x)-\phi(x)|:x\in\gamma$ – *Microm* (0). Let x be any element in γ –*Microm* (0) and apply the mean value theorem with η being an upper bound of $\frac{\partial f}{\partial y}$, then

$$\left|\psi(x)-\phi(x)\right| \le \int \eta\left|\psi(x)-\phi(x)\right|ds$$

$$=\le \ |x|\,m \tag{4}$$

Since $x\in\gamma$ –*Microm* (0), then $|\psi(x)-\phi(x)|$ will become <ξ, where ξ is an infinitesimal, which contradicts the inequality (3). This completes the proof.

IV. Legendre Equation

The differential equation of Legendre is of the form:

$$\left(1-x\ \right)\frac{d\ y}{dx}-2x\frac{dy}{dx}+l\left(l+1\right)y=0, \tag{5}$$

Having the parameter l [12]. Now, we will try to find the solution infinitely closed to the singularity, where x is in the monad of zero. However, the singular points of equation (5) are $x = \omega$, $x = 1$, and $x = -1$, where ω is unlimited. These singular points are regular singular points. Thus, use a suitable transformation to make the singularity in the monad of zero.

Now, since ω is a regular singular point, and — is infinitesimal. Thus, let __. The derivatives with respect to x becomes

$$\frac{dy}{dx}=\frac{d\zeta}{dx}\frac{dy}{d\zeta}=\frac{-1}{x^2}\frac{dy}{d\zeta}=-\zeta^2\frac{dy}{d\zeta},$$

$$\frac{d^2y}{dx^2}=\frac{d\zeta}{dx}\frac{d}{d\zeta}\left(\frac{dy}{dx}\right)$$

$$=-\zeta^2\left(-2\zeta\frac{dy}{d\zeta}-\zeta^2\frac{d^2y}{d\zeta^2}\right)$$

$$=\zeta^3\left(2\frac{dy}{d\zeta}+\zeta\frac{d^2y}{d\zeta^2}\right).$$

If we substitute these derivatives of equation (2) into the Legendre's equation (5), we get

$$\left(1-\frac{1}{\zeta^2}\right)\zeta^3\left(2\frac{dy}{d\zeta}+\zeta\frac{d^2y}{d\zeta^2}\right)+2\left(\frac{1}{\zeta}\right)\zeta^2\frac{dy}{d\zeta}+l\left(l+1\right)y=0.$$

Then, equation (1) will become

$$\zeta^2(\zeta^2-1)\frac{d^2y}{d\zeta^2}+2\zeta^3\frac{dy}{d\zeta}+l(l+1)y=0. \tag{6}$$

Since $\zeta\in m(0)$ is a regular singular point, then the method of Frobenius advises us that we can write an approximate power series solution for the equation (5) of the form:

$$y\cong\sum_{i=0}^{\omega_1}c_i\zeta^{i+r}, \tag{7}$$

Where ω_1 is unlimited positive even integer. Put equation (7) in equation (6), we get

$$\zeta^r\sum_{i=0}^{\omega_1}c_i\left(\left(r+i+l\right)\left(r+i-l-1\right)\zeta^i+\left(r+i\right)\left(r+i+1\right)\zeta^{i+2}=0.\right.$$

Hence,

$c_0(r+l)(r-l-1) = 0.$

By assumption $c_0\neq0$, then $r = -l$ or $r = l+1$, and

$$c_{i+2}=\frac{(r-i)(r+i+1)}{(r+l+i+2)(r+l-i-1)}c_i, \quad i\ge0.$$

First, for $r = l+1$, then $c_1(l+1) = 0$ or $c_1 = 0$, and

$$c_{i+2}=-\frac{(l+i+1)(l+i+2)}{(i+2)(2l+i+3)}c_i,$$

So that $c_{2i} = 0$ for $i\ge0$. Therefore,

$$y_1\cong c_1\zeta^{l+1}\left(1+\frac{(l+1)(l+2)}{2(2l+3)}\zeta^2+\frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot4\cdot(2l+3)(2l+5)}\zeta^4+\frac{(l+1)(l+2)(l+3)(l+4)\cdots(l+\omega_1)}{2\cdot4\ldots\omega_1\cdot(2l+3)(2l+5)\cdots(2l+\omega_1+1)}\zeta^{\omega_1}\right),$$

Moreover, it is true when $\zeta\in m(0)$. Then,

$$y_1\cong c_1x^{-l-1}\left(1+\frac{(l+1)(l+2)}{2(2l+3)}x^{-2}+\frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot4\cdot(2l+3)(2l+5)}x^{-4}+\frac{(l+1)(l+2)(l+3)(l+4)\cdots(l+\omega_1)}{2\cdot4\ldots\omega_1\cdot(2l+3)(2l+5)\cdots(2l+\omega_1+1)}x^{-\omega_1}\right),$$

Hence, y_1 is a solution, whenever x is unlimited.

Second, for $r = -l$, then $c_1 l = 0$. That is $c_1 = 0$, and

$$c_{i+2}=-\frac{(l-i)(l-i-1)}{(i+2)(2l-i-1)}ci,$$

DOI: http://dx.doi.org/10.14500/icpas2018

Giving $c_{2i+1} = 0$ for $i \geq 0$, and

$$c_2 = -\frac{l(l-1)}{2(2l-1)}\,c0,$$

$$c_4 = \frac{l(l-1)(l-2)(l-3)}{2\cdot 4\cdot (2l-1)(2l-3)}\,c0.$$

In general, we have

$$c_{\omega_1} = \frac{(l)(l-1)(l-2)(l-3)\cdots(l-(\omega_1-1))}{2\cdot 4\cdots \omega_1\cdot (2l-1)(2l-3)\cdots(2l-(\omega_1-1))}\,c0.$$

Thus,

$$y_2 \cong c_0\zeta^{-1}\Big(1-\frac{l(l-1)}{2(2l-1)}\zeta^2+\frac{l(l-1)(l-2)(l-3)}{2\cdot 4\cdot(2l-1)(2l-3)}\zeta^4 -$$

$$+(-1)^{\omega_1}\frac{(l)(l-1)(l-2)(l-3)\cdots(l-(\omega_1-1))}{2\cdot 4\cdots\omega_1\cdot(2l-1)(2l-3)\cdots(2l-(\omega_1-1))}\zeta^{\omega_1}\Big),$$

Where $\zeta \in m(0)$, and $x = \frac{1}{\zeta}$.

$$y_2 \cong c_0 x^1\Big(1-\frac{l(l-1)}{2(2l-1)}x^{-2}+\frac{l(l-1)(l-2)(l-3)}{2\cdot 4\cdot(2l-1)(2l-3)}x^{-4} -$$

$$+(-1)^{\omega_1}\frac{(l)(l-1)(l-2)(l-3)\cdots(l-(\omega_1-1))}{2\cdot 4\cdots\omega_1\cdot(2l-1)(2l-3)\cdots(2l-(\omega_1-1))}x^{-\omega_1}\Big).$$

Hence, y_2 is a solution, whenever x is unlimited.

For $x = 1$ is a regular singular point for the Legendre equation(1). We exchange the variable x by ξ, to make the regular singular point in the monad of zero. Hence, let $\xi = x-1$, then we get

$$(2+\xi)\frac{d^2y}{d\xi^2}-2(1+\xi)\frac{dy}{d\xi}+l(l+1)y=0. \tag{8}$$

So, the regular singular point is $\xi = 0$. Hence, we can find the approximate solutions for equation (8) using the Frobenius method such as equation (4). Similarly, we can find approximate solutions for equation (5) in the monad of $x = -1$.

Now, in the solution of equation (6) if we take l as an appreciable natural number and $\frac{1}{x} \in m(0) \cap (\zeta - Microm(0))_c$ whenever ζ is an infinitesimal, then the solution y_1 becomes limited for positive limited c_1. However, there are no solutions y_2 which approach to a limited number whenever c_0 approaches to an appreciable or unlimited number except the case that if we take $c_0 \in \zeta - Microm(0)$.

By Mehler–Dirichlet Integral [13], we can write Legendre polynomial as follows:

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin((n+\frac{1}{2})t)}{\sqrt{2(\cos\theta-\cos t)}}dt, 0<\theta<\pi,\ n=0,1,2, \tag{9}$$

1) If $t \cong 0$, then sin $((n+\frac{1}{2})t) \cong t$ for limited n and $cos(t) \cong 1$ [14]. Therefore,

$$Pn(\cos\theta) \cong \frac{2}{\pi}\int_\theta^\pi \frac{t}{\sqrt{2(\cos\theta-1)}}dt$$

$$\cong \frac{(\pi^2-\theta^2)}{\pi\sqrt{2(\cos\theta-1)}}.$$

2) If $t \cong \pi$, then there exists an infinitesimal ξ such that $t = \pi - \xi$, then $dt = -d\xi$, and if $t = \theta$ then $\xi = \pi - \theta$, also, if $t = \pi$ then $\xi = 0$. Hence,

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^{\pi-\theta}\frac{\sin((n+\frac{1}{2})(\pi-\xi))}{\sqrt{2(\cos\theta-\cos(\pi-\xi))}}d\xi$$

$$= \frac{2}{\pi}\int_0^{\pi-\theta}\frac{\sin((n+\frac{1}{2})\pi)\cos((n+\frac{1}{2})\xi)}{\sqrt{2(\cos\theta-\cos(\pi-\xi))}}d\xi.$$

Since $\xi \cong 0$, this implies that $cos\xi \cong 1$, $sin((n+\frac{1}{2})\pi) = (-1)^n$. Therefore,

$$P_n(\cos\theta) \cong \frac{2}{\pi}\int_0^{\pi-\theta}\frac{(-1)^n}{\sqrt{2(\cos\theta+1)}}d\xi.$$

Thus,

$$P_n(\cos\theta) \cong \frac{2(-1)^n(\pi-\theta)}{\pi\sqrt{2(\cos\theta+1)}}.$$

V. Conclusion

With this study, we conclude that the nonstandard tools were the most powerful technique to overcome the difficulty and inability for obtaining approximate solution and the behavior of solution curves near singularities and guaranteed of existence of unique infinitely near solution of the problems given by mathematical models of real applications represented by differential equations.

VI. Acknowledgment

The authors would like to thank the referees, for their valuable comments and suggestions.

References

[1] M. Müller, "Über das fundamentaltheorem in der theorie der gewöhnlichen differentialgleichungen," *Mathematische Zeitschrift*, vol. 26, no. 1, pp. 619-645, 1927.

[2] E. Kamke, "Zur theorie der systeme gewöhnlicher differentialgleichungen. II, *Acta Mathematica*, vol. 58.1, pp. 57-85, 1932.

[3] W. Walter, "Ordinary differential inequalities in ordered Banach spaces," *Journal of Differential Equations,* vol. 9, no. 2, pp. 253-261, 1971.

[4] E. Nelson, *Radically Elementary Probability Theory*, Princeton, New Jersey: Princeton University Press, p. 16, 1987.

[5] F. Diener and M. Diener, *Nonstandard Analysis in Practice*, Berlin, Heildeberg: Springer-Verlag, 1995.

[6] A. Robinson, *Non-Standard Analysis*, Los Angeles: University of California, 1966.

[7] R. Goldblatt, *Lectures on the Hyperreals an Introduction to Nonstandard Analysis*, New York: Springer Verlag, pp. 49-50, 1998.

DOI: http://dx.doi.org/10.14500/icpas2018

[8] E. Nelson, “Internal et theory, a new approach to nonstandard analysis,” *Bulletin of the American Mathematical Society,* vol. 83, no. 6, pp. 1165-1198, 1977.

[9] I. Goldbring, *Lecture Notes on Nonstandard Analysis*, Chicago: University of Chicago Press, 2014.

[10] R. Lutz and M. Goze, Non Standard Analysis: A Practical Guide with Applications, Lectures Notes in Math. vol. 881. Berlin: Springer-Verlag, 1982.

[11] V. I. Arnold, In R. A. Silverman, Ed. *Ordinary Differential Equations*, Cambridge: MIT Press, 1973.

[12] B. L. J. Braaksma and B. Meulenbeld, “Integral transforms with generalized Legendre functions as kernels,” *Compositio Mathematica*, vol. 18, no. 3, pp. 235-287, 1967.

[13] A. D. Poularikas, Ed, *Handbook of Formulas and Tables for Signal Processing*, vol. 13. Boca Raton: CRC Press, 1998.

[14] I. O. Hamad, “A Nonstandard Asymptotic Approximation of Bessel Function,” Special Issue of 2nd International Mathematics Conference of Zarqa Private University Jordon, pp. 244-257, 2008.

DOI: http://dx.doi.org/10.14500/icpas2018