Weak Idempotent Elements

Rebin M. Hassan[1] and Parween A. Hummadi[2]

[1]Department of Mathematics, University of Raparin, Main Road, Rania/Al-Sulaimania, Kurdistan Region, Iraq

[2]Department of Mathematics, University of Salahaddin, Kirkuk Road, Hawler (Erbil), Kurdistan Region, Iraq

***Abstract*—In this work we introduce the concept of weak idempotent element as a generalization of the concept of the idempotent element and to investigate weak idempotents in Z_n and in some type of group rings. It is shown that if n is square free integer, then every element of Z_n is weak idempotent. Characterization of a weak idempotent in $Z_{2^n}G$ where G is a cyclic group of order 2 is given. Furthermore, it is proved that an element of Z_2G where G is a cyclic group of order 2^n is weak idempotent if and only if it has an odd number of summands.**

***Index Terms*—Idempotent element, Nilpotent element, Ring, Weak idempotent element.**

I. Introduction

Given a binary operation $*$ on a set S, an element x is said to be idempotent if $x*x=x$. An element of a ring is idempotent if $x^2=x$, idempotent elements have an important role in a decomposition of rings. There are rings in which every element is idempotent (Boolean ring). In 2010 the concept of m-idempotent element introduced and studied in Huang and Guo [1] as a generalization of an idempotent element. An element x of a ring R is m-idempotent if m is the least positive integer such that $x^m=x$. In this paper, the concept of weak idempotent element introduced as a generalization of the idempotent element. An element of a ring R is called weak idempotent if x^n-x is idempotent. Such element studied in this work especially in Z_n, the ring of integers modulo n and in some type of group rings.

II. Preliminaries

In this section, we state some lemmas needed in this work, we start by the following lemma.

Lemma 1. If p and q are distinct primes and x is divisible by p, then $x^q \equiv x \ (mod\ pq)$.

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Corresponding author's e-mail: rebin.srwsht@gmail.com

Proof: The case when x is divisible by q, is trivial, so suppose that x is not divisible by q. Then by Fermat's Theorem [2, p. 165], $x^{q-1}\equiv 1(mod\ q)$, that is, $x^{q-1}-1$ is divisible by q. Thus, x^q-x is divisible by pq. Hence, $x^q-x\equiv 0(mod\ pq)$, consequently $x^q\equiv x(mod\ pq)$.

Lemma 2 [3, p. 140]. If x is an odd number, then $x^2\equiv x$ $(mod\ 2x)$.

Lemma 3. If p is an odd prime, then $a^p\equiv a(mod\ 2p)$, for each $a\in Z_{2p}$.

Proof. If $gcd(a,2p)=1$, then by Euler's Theorem, $a^{p-1}\equiv 1(mod\ 2p)$, consequently $a^p\equiv a(mod\ 2p)$. Now, suppose that $gcd(a,2p)>1$, then $gcd(a,2p)=p$ or $gcd(a,2p)=2$. In case $gcd(a,2p)=p$, the only possibility is $a=p$. Then, $a^2 = p^2 = \underbrace{p+p+\cdots+p}_{p-times} \equiv p(mod\ 2p)$.

Hence, $p^3=pp^2\equiv p(mod\ 2p)$, continuing in this manner, we get $a^p=p^p\equiv p(mod\ 2p)$. In case $gcd(a,2p)=2$, that is $a=2^nl$, for some positive integer $n\geq 1$ and $gcd(l,2p)=1$, we have $a^p=(2^nl)^p\equiv 2^nl(mod\ p)$, so $p|(2^nl)^p-2^nl$, but $2|(2^nl)^p-2^nl$, hence $2p|(2^nl)^p-2^nl$. This implies $(2^nl)^p\equiv 2^nl(mod2p)$. Thus $a^p\equiv a(mod2p)$.

Lemma 4. Let a_0+a_1g be an element of the group ring $Z_{2p}G$, where is a cyclic group of order 2 generated by g, and p is an odd prime. If a_0 and a_0 are odd modulo $2p$, then a_0+a_1g is not idempotent.

Lemma 5 [4, p. 26]. Let n be a fixed and a, b be arbitrary integers. If $a\equiv b(modn)$, then $a^k\equiv b^k(mod\ n)$, for any positive integer k.

Lemma 6 [5, p. 29]. In the group ring Z_pG, where p is prime and G is a cyclic group of order q, $(a_0+a_1g+a_2g^2+\ldots+a_{q-1}g^{q-1})^p=a_0^p+(a_1g)^p+\ldots+(a_{q-1}g^{q-1})^p$.

Lemma 7. Let A be an element of $M_{2\times 2}(Z_p)$, such that $A^2=bA$, $b\in Z_p$, then $A^n=b^{n-1}A$, for each $n\in Z^+$.

Proof. We use the induction on n. If $n=2$, then by assumption $A^2=bA$. Now, suppose that it is true for n=k, which means that $A^k=b^{k-1}A$ Since $A^{k+1}=A^kA$, then $A^{k+1}=b^{k-1}AA=b^{k-1}A^2$, so $A^{k+1}=b^{k-1}bA=b^kA$. Hence $A^{k+1}=b^kA$.

III. Weak Idempotents

In this section, we study weak idempotent elements in Z_n and in the group ring Z_nG, where G is a cyclic group of finite order. Clearly, every idempotent is weak idempotent.

Proposition 1. Every element of Z_p, (p is prime) is a weak idempotent element.

Proof. By Fermat's Theorem [2, p. 165], $x^p\equiv x(mod\ p)$, for each x in Z_p. Then $x^p-x\equiv 0(mod\ p)$, hence x is a weak idempotent element.

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Proposition 2. Every element in Z_{pq} is a weak idempotent element, where p and q are distinct primes.

Proof. Let $x \in Z_{pq}$. If x is neither divisible by p nor by q, then by Euler's Theorem [4, p. 26], we have $x^{\varphi(pq)} \equiv 1(mod\ pq)$, that is $x^{(p-1)(q-1)} \equiv 1(mod\ pq)$, consequently $x^{(p-1)(q-1)+1} \equiv x(mod\ pq)$, so x is weak idempotent. If x is divisible by p or q, say it is divisible by p, then by Lemma 1, we have $x^q \equiv x(mod\ pq)$. Hence, x is a weak idempotent element.

Theorem 3. A nontrivial nilpotent element in a ring R, cannot be weak idempotent.

Proof. Let $0 \neq x$ be a nilpotent element of a ring R and n be the least positive integer such that $x^n=0$. Suppose that $(x^l-x)^2=x^l-x$…(1), for some $l>1$.

If $l>n$, then clearly $x^l=x^{n+m}=x^n.x^m=0$ … (2) for some m in Z^+. So by substituting (2) in (1), we get $x^2+x=0$. Multiplying both sides by x^{n-2} we get $x^{n-1}=0$, which is a contradiction with n is the least positive integer such that $x^n=0$. If $l=n$, then $x^l=0$, and similarly we get a contradiction.

If $l<n$, then from (1) we get $x^{2l}-2x^{l+1}+x^2=x^l-x$. Multiplying both sides by x^{n-2} we get $x^{n+2l-2}-2x^{n+l-1}+x^n=x^{n+l-2}-x^{n-1}$, hence $x^{n-1}=0$, which is also a contradiction. Thus, $(x^l-x)^2 \neq x^l-x$ for each l>1. Hence, x cannot be a weak idempotent element.

Proposition 4. In the ring Z_{p^n}, an element x is a weak idempotent element if and only if x is relatively prime to p.

Proof. Suppose that $gcd(x,p)=1$, equivalently $gcd(x,p^n)=1$. Then by Euler's Theorem, $x^{\varphi(p^n)} \equiv 1\ (mod\ p^n)$, so $x^{(\varphi(p^n)+1)} - x \equiv 0(mod p^n)$ hence, x is a weak idempotent element. Now, if $x \equiv kp(\text{mod } p^n)$, for some positive integer k then $x^n \equiv 0(mod\ p^n)$, so x is a nilpotent element, and by **Theorem 3,** x cannot be weak idempotent.

Proposition 5. Consider Z_n, with the prime factorization of $n = p_1^{t_1}\, p_2^{t_2} \cdots pk^{t_k}$, with at least one of $t_i>1$, and let x be a non-zero element of Z_n. If $x = q_1^{s_1}\, q_2^{s_2} \cdots q_\ell^{s_\ell}$ with $l<k$ and satisfies

i. $gcd(m,n)=1$,
ii. For each i there is j with $i \leq j \leq k$ such that $q_i=p_j$.
iii. Either $q_i \nmid x$, for each $1 \leq i \leq l$ or $s_i \geq t_i$ for each $1 \leq i \leq l$, then x is a weak idempotent element.

The proof of proposition 5 is similar to the proof of Theorem 2.2.10 of Ali [5].

Proposition 6. Let R_i, $1 \leq i \leq k$ be a ring with 1 and $R=R_1 \times R_2 \times \ldots \times R_k$.

If $x=(a_1, a_2, \ldots, a_k) \in R$ and for each $1 \leq i \leq k$, $a_i^{n_i}=1$, for some $n_i>0$, then x is a weak idempotent element.

Proof. Let $x=(a_1, a_2, \ldots, a_k) \in R$, where $a_1^{n_1}=1, a_2^{n_2}=1, \ldots a_k^{n_k}=1$, and let $m=lcm(n_1, n_2, \ldots, n_k)$. Now, $x^m=(a^1, a^2, \ldots, a_k)^m$, $x^m = \left(a_1{}^m, a_2{}^m, \ldots, a_k{}^m\right)$. Then $x^m=(1, 1, \ldots, 1)$, so $x^{m+1}=(a_1, a_2, \ldots, a_k)=x$. Hence, is a weak idempotent element.

Proposition 7. Every element of the ring $Z_{p_1 p_2 \ldots p_k}$ is a weak idempotent element, where $p_1, p_2, \ldots, p_k$ are distinct primes.

Proof. It is well known that $Z_{p_1 p_2 \ldots p_k} \cong Z_{p_1} \times Z_{p_2} \times \ldots \times Z_{p_n}$ [6, p. 36], and since for each $a_i \in Z_{p_i}$, $a_i^{p_i-1} \equiv 1(mod\ p_i)$, then by **Proposition 6**, every element of $Z_{p_1} \times Z_{p_2} \times \ldots \times Z_{p_n}$ is a weak idempotent element, then every element of $Z_{p_1 p_2 \ldots p_k}$ is a weak idempotent element.

In what follows, we study weak idempotent elements in some type of group rings.

Proposition 8. Every element in the group ring Z_pG, where G is a cyclic group of order 2 generated by g and p is an odd prime, is a weak idempotent element.

Proof. Let $x=a_0+a_1g \in Z_pG$. Then $x^p=(a_0+a_1g)^p$, and by **Lemma 6**,

$x^p=(a_0)^p+(a_1g)^p=(a_0)^p+(a_1)^pg$, then by Fermat's Theorem, $x^p=a_0+a_1g=x$, thus $x^p-x=0$, hence x is weak idempotent.

Note that proposition 9, is not true for $p=2$, since $1+g$ is a nilpotent element in Z_2G and by **Theorem 3, 1+*g*** cannot be a weak idempotent element.

Theorem 9. Let $Z_{2p}G$ be a group ring of G over Z_{2p}, where G is a cyclic group of order 2 generated by g and p is an odd prime. Then, an element a_0+a_1g of Z_{2p} is weak idempotent if and only if at least one of a_0, a_1 is even.

Proof. Let $x=a_0+a_1g \in Z_{2p}G$, and suppose that $a_0=2m \in Z_{2p}$, for some positive integer m. We claim that $x^p=x$ in $Z_{2p}G$.

Now, $$\begin{aligned} x^p &= (2m+a_1g)^p = (2m)^p + p(2m)^{p-1}(a_1g) \\ &+ \frac{p(p-1)}{2!}(2m)^{p-2}(a_1g)^2 + \ldots + (a_1g)^p \end{aligned}$$, so by **Lemma 3**, $(2m)^p \equiv 2m$ $(mod\ 2p)$, and $a_1{}^p \equiv a_1(mod\ 2p)$, then $x^p=2m+a_1g=x$. Now, suppose that $gcd(a_1, 2p)>1$, then $gcd(a_1, 2)=2$ or $a_1=p$. If $gcd(a_1, 2)=2$, then by **Lemma 3**, $a_1{}^p \equiv a_1\left(mod\ 2p\right)$, hence $x^p=x$. In both cases, x is weak idempotent.

Conversely, suppose that both a_0 and a_1 are odd numbers. We have the following cases.

Case (1): If $x=p+pg$, then $x^2=2\rho^2+2\rho^2g=0$, then x is a nilpotent element. Hence, by **Theorem 3**, x cannot be weak idempotent.

Case (2): If $x=p+a_1g$, where $a_1 \neq p$, then $x^2 = p^2+a_1{}^2$, and by **Lemma 2**, $p^2 \equiv p(mod\ 2p)$, so . We claim that $x^n \neq x$ for each n. If n is even, then $x^n = p + a_1{}^n \neq x$.

If n is odd, then for $k \in Z^+$, $x^n = x^{2k+1} = x^{2k}$. $x = \left(p + a_1{}^{2k}\right)\left(p + a_1g\right)$, since $\left(p + a_1{}^{2k}\right)$ is even, so $\left(p + a_1{}^{2k}\right)p \equiv 0\left(mod\ 2p\right)$. Thus $x^n = \left(p + a_1{}^{2k}\right)a_1g \neq x$.

Now, $$\begin{aligned} \left(x^n - x\right)^2 &= \left(\left(p + a_1{}^{2k}\right)a_1g - (p + a_1g)\right)^2 \\ &= \left(p + \left(p + a_1{}^{2k} - 1\right)a_1g\right)^2 \\ &= p^2 + \left(\left(p + a_1{}^{2k} - 1\right)a_1\right)^2 \neq x^n - x \end{aligned}$$. Hence, x is not weak idempotent.

Case (3): If $x=a_0+a_1g$, where both a_0 and a_1 are different from p. Then $x^2 = a_0{}^2 + a_1{}^2 + 2a_0a_1g \neq x$, as $a_0{}^2 + a_1{}^2$

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and $2a_0a_1$ are even. Put $a_0{}^2 + a_1{}^2 \equiv 2l(mod\, 2p)$ and $2a_0a_1 \equiv 2_k(mod\ 2p)$, so $x^2=2l+2k$. Now, for each $n>2$, $x^n=x^{n-2}$.

$x^2=(b_0+b_1g)(2l+2kg)$
$=2(lb_0+kb_1)+2(kb_0+lb_1)g\neq x,$

Where b_0, $b_1\in Z_2p$, and by **Lemma 4**, x^n-x is not idempotent. Hence, is not weak idempotent.

Proposition 10. Let $Z_{pq}G$ be the group ring of G over Z_{pq}, where G is a cyclic group of order 2 generated by g and p, q are distinct odd primes. If x has one of the forms $x=a_0$, $x=a_1g$, $x=m_0p+k_0pg$, and $x=m_1q+k_1qg$, then x is a weak idempotent element.

Proof. If $x=a_0$, then by Proposition 2, x is weak idempotent. If $x=a_1g$, then as in Proposition 2, if a_1 neither divisible by p nor q, then $(a_1)^{(p-1)(q-1)+1}\equiv a_1(mod\ pq)$. Thus, $(x)^{(p-1)(q-1)+1}=(a_1)^{(p-1)(q-1)+1}g^{(p-1)(q-1)+1}=a_1g=x$. Hence, is weak idempotent. If a_1 is divisible by p or q, say it is divisible by p, then $(a_1)^q\equiv a_1(mod\ q)$. So $(x)^q=a_1{}^qg^q=a_1g=x$. Hence, x is weak idempotent. If $x=m_0p+k_0pg$, then

$$x^q = (m_1p)^q + q(m_1p)^{q-1}(k_1pg) + \frac{q(q-1)}{2!}(m_1p)^{q-2}(k_1pg)^2 + \ldots + (k_1pg)^q.$$

So $x^q=(m_1p)^q+(k_1pg)^q$, then by Lemma 1, $x^q=m_1p+k_1pg=x$. Similarly, if $x=m_1q+k_1qg$, then $x^p=x$. Hence, x is a weak idempotent element.

Proposition 11. Let Z_2G be the group ring of G over Z_2, where G is a cyclic group of order n generated by g. Then, g^k is a weak idempotent element, where $1\leq k\leq n-1$.

Proof. Put $x=g^k$. Then, $x^n=\left(g^k\right)^n=1$, so $x^{n+1}-x=0$. Hence, is a weak idempotent element.

Proposition 12. Let Z_2G be the group ring of G over Z_2, where is a cyclic group of order 2^n generated by g. Then, an element x of Z_2G is a weak idempotent element if and only if x has an odd number of summands.

Proof. Suppose that x has an odd number of summands, then either $x=g^{i_1}+g^{i_2}+\ldots+g^{i_k}$, where $1\leq i_j\leq 2^n-1$ and k is odd, or $x=1+g^{i_1}+g^{i_2}+\ldots+g^{i_k}$, where $1\leq i_j\leq 2^n-1$ and k is even. In the first case $x^{2^n}=\underbrace{1+1+\cdots+1}_{k-times}=k\equiv 1\ (mod\, 2)$, in the second case $x^{2^n}=k+1\equiv 1\ (mod\, 2)$.

In both cases we get $x^{2^n+1}-x=0$. Hence, x is a weak idempotent element. Conversely, let x be a weak idempotent element, we will show that x has an odd number of summands. Suppose that x has an even number of summands.

Then either $x=g^{i_1}+g^{i_2}+\ldots+g^{i_k}$, where $1\leq i_j\leq 2^n-1$ and k is even, or $x=1+g^{i_1}+g^{i_2}+\ldots+g^{i_k}$, where $1\leq i_j\leq 2^n-1$ and k is odd. As before, in both cases we get $x^{2^n}\equiv 0(mod\, 2)$, so x is a nilpotent element, then by **Theorem 3**, x cannot be weak idempotent, which is a contradiction with assumption. Hence, x has an odd number of summands.

Remark 13. Let a_0+a_1g be an element of the group ring $Z_{2^n}G$, where G is a cyclic group of order 2 generated by . Then

$$(a_0+a_1g)^{2^n} = a_0{}^{2^n} + \frac{2^n(2^n-1)}{2!}(a_0)^{2^n-2}(a_1g)^2 + \frac{2^n(2^n-1)(2^n-2)(2^n-3)}{4!}a_0{}^{2^n-4}(a_1g)^4 + \ldots + \frac{2^n(2^n-1)}{2!}a_0{}^2(a_1g)^{2^n-2} + (a_1g)^{2^n}$$

If a_0 is odd and a_1 is even, then clearly $(a_0+a_1g)^{2^n}=a_0{}^{2^n}$. If a_0 is even and a_1 is odd, then similarly $(a_0+a_1g)^{2^n}=(a_1g)^{2^n}=a_1{}^{2^n}$. If a_0 and a_1 are even, then $(a_0+a_1g)^{2^n}=0$.

Theorem 14. Let $Z_{2^n}G$ be the group ring of G over Z_{2^n}, where G is a cyclic group of order 2 generated by g. Then, a_0+a_1g is a weak idempotent element if and only if one of a_0, a_1 is even and the other is odd modulo 2^n.

Proof. Let $x=a_0+a_1g\in Z_{2^n}G$. Suppose that a_1 is even and a_0 is odd. We will show that $x^{2^n+1}=x$. Since $gcd(a_0,2^n)=1$, then by Euler's Theorem, $a_0{}^{\varphi(2^n)}\equiv 1\ (mod\, 2^n)$, so $a_0{}^{2^{n-1}}\equiv 1\ (mod\, 2^n)$, then by **Lemma 5**, $a_0{}^{2^n}\equiv 1(mod\, 2^n)$. Now $x^{2^n}=(a_0+a_1g)^{2^n}$, then by **Remark 13**, $x^{2^n}=a_0{}^{2^n}$, so $x^{2^n}=1$, then $x^{2^n+1}=x$. Hence, x is weak idempotent.

The case when a_0 is even and a_1 is odd can be proved similarly.

For the converse, suppose that both a_0 and a_1 are even numbers. Then by **Remark 13**, $x^{2^n}=a_0{}^{2^n}\equiv 0\ (mod\, 2^n)$, then x is a nilpotent element. Hence, by **Theorem 3**, x cannot be a weak idempotent element. Now, suppose that both a_0 and a_1 are odd numbers. Then $x^2=a_0{}^2+a_1{}^2+2a_0a_1g\neq x$. Since both $a_0{}^2+a_1{}^2$ and $2a_0a_1$ are even, then by **Remark 13**, $(x^2)^{2^n}=0$, then x is a nilpotent element. Hence, by **Theorem 3**, x cannot be weak idempotent.

Finally, we study weak idempotent elements in $M_{2\times 2}(Z_p)$, where p is a prime number.

Theorem 15:[1(1)] Let $A=\begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}\in M_{2\times 2}(Z_p)$, where p is prime and $M_{2\times 2}(Z_p)$ is the ring of all 2×2 matrices over the ring Z_p, then:

1. If A is a singular matrix and $c_0+c_3\equiv 0(mod\ p)$, then A cannot be weak idempotent
2. If A is a singular matrix and $c_0+c_3\not\equiv 0(mod\ p)$, then A is weak idempotent
3. If A is a nonsingular matrix and $c_0+c_3\equiv 0(mod\ p)$, then A is weak idempotent.

Proof.

1. Suppose that $A=\begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}\in M_{2\times 2}(Z_p)$ is a singular matrix and $c_0+c_3\equiv 0(mod\ p)$. Then $c_0c_3\equiv c_1c_2(mod\ p)$.

1 We use the GAP program given in the appendix to find weak idempotents in matrix rings.

Now,

$$A^2=\begin{pmatrix} c_0^{\ 2}+c_1c_2 & c_0c_1+c_1c_3 \\ c_2c_0+c_2c_3 & c_3^{\ 2}+c_1c_2 \end{pmatrix}=\begin{pmatrix} c_0^{\ 2}+c_0c_3 & c_1(c_0+c_3) \\ c_2(c_0+c_3) & c_3^{\ 2}+c_0c_3 \end{pmatrix}$$

$$=\begin{pmatrix} c_0(c_0+c_3) & c_1(c_0+c_3) \\ c_2(c_0+c_3) & c_3(c_0+c_3) \end{pmatrix}$$

So $A^2=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which means that A is a nilpotent element. Hence, by **Theorem 3**, A cannot be a weak idempotent element.

2. Suppose that A is singular and $c_0+c_3\not\equiv 0(mod\ p)$. Then

   $A^2=\begin{pmatrix} c_0(c_0+c_3) & c_1(c_0+c_3) \\ c_2(c_0+c_3) & c_3(c_0+c_3) \end{pmatrix}$. If $c_0+c_3\equiv 1(mod\ p)$, then

   $A^2=A$, hence A is weak idempotent. If , put $b\equiv c_0+c_3(mod\ p)$, $b\in Z_p$ and $b\notin\{0,1\}$, so $A^2=bA$, then by **Lemma 7** $A^p=b^{p-1}A$, hence by Fermat's Theorem, $A^p=A$. Thus, A is a weak idempotent element.

3. Suppose that A is nonsingular and $c_0+c_3\equiv 0(mod\ p)$. Since

$$A^2=\begin{pmatrix} c_0^{\ 2}+c_1c_2 & c_1(c_0+c_3) \\ c_2(c_0+c_3) & c_3^{\ 2}+c_1c_2 \end{pmatrix}, \text{ then}$$

$$A^2=\begin{pmatrix} c_0^{\ 2}+c_1c_2 & 0 \\ 0 & c_3^{\ 2}+c_1c_2 \end{pmatrix}.$$

Put $a\equiv c_0^{\ 2}+c_1c_2(mod\ p)$, and $d\equiv c_3^{\ 2}+c_1c_2(mod\ p)$, so $A^2=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

(Note that $(A^2)^n=\begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix}$, for each positive integer n.)

Now, $(A^2)^{p-1}=\begin{pmatrix} a^{p-1} & 0 \\ 0 & d^{p-1} \end{pmatrix}$, then by Fermat's Theorem,

$A^{2(p-1)}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, hence $\begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}=A^{2(p-1)+1}=A$. Thus, A is weak idempotent.

Corollary 16. Let $M_{2\times 2}\ (R)$ be the ring of all 2×2 matrices over the ring R. If every element of R is a weak idempotent element, then an element of $M_{2\times 2}\ (R)$, need not be a weak idempotent element.

Example 17. Let $R=Z_5$, then by **Proposition 1**, every element of R is a weak idempotent element. Since $A=\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\in M_{2\times 2}(R)$ is a nilpotent element, then by **Theorem 3**, A cannot be weak idempotent.

IV. Conclusion

In the following, some obtained results of this work are stated.

1. A nontrivial nilpotent element in a ring R, cannot be weak idempotent.
2. Every element in the group ring Z_pG, where is a cyclic group of order 2 generated by g and p is an odd prime, is a weak idempotent element.
3. Let Z_2pG be a group ring of G over Z_2p, where G is a cyclic group of order 2 generated by g and p is an odd prime. Then, an element a_0+a_1g of Z_2p is weak idempotent if and only if at least one of a_0, a_1 is even.
4. Let Z_2G be the group ring of G over Z_2, where G is a cyclic group of order 2^n generated by g. Then, an element x of Z_2G is a weak idempotent element if and only if x has an odd number of summands.
5. Let $Z_{2^n}G$ be the group ring of G over Z_{2^n}, where G is a cyclic group of order 2 generated by g. Then, a_0+a_1g is a weak idempotent element if and only if one of a_0, a_1 is even and the other is odd modulo 2^n.

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APPENDIX

The following (GAP)-code (Computer Algebra System) is used to verify the results of this work.

```
# weak idempotent elements of matrix#
GR:=MatAlgebra(ZmodnZ(3),2);
A:=[];
B:=[];
e:=Elements(GR);
  for i in [1. Size(e)] do
    for j in [2. Size(e)] do
    x:=e[i];
    if (x^j-x)^2=x^j-x then
     Add(A,x);
    fi;
  od;
  if not x in A then
    Add(B,x);
  fi;
od;
A:=AsSet(A);
B:=AsSet(B);
Print("The set of weak idempotent elements is ",A,"\n");
Print("The set of elements which are not weak idempotent elements is ",B);
```

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